Determining Optimal Policy for the Federal Reserve

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Abstract

In this paper, we explore optimal monetary policies to drive an economy towards a desirable rate of inflation and level of economic output. We set up an optimization problem that minimizes inflation and the differences between efficient economic output and realized economic output. Our control variable will be the interest rate mandated by the Federal Reserve. Using Pontryagin's Maximum Principle, we attempt to find the optimal interest rate policy that will combat inflation while stimulating economic output under various conditions (for both finite time periods and infinite horizons). After achieving our results regarding optimal monetary policies, we briefly address the extension of our work in the realm of governmental fiscal decisions. We find that while our results with regard to monetary policy are fairly convincing, much future work remains to be done for the government spending problem.

1 Background

The 2008 Great Recession featured several systematic shocks that presented major challenges in restoring stability to the United States economy. One such shock was to the "natural interest rate". That is, investors grew their risk aversion to the point where they were unwilling to purchase interest-free bonds. This limited the Federal Reserve's ability to stimulate economic activity via a nominal interest rate. Some central banks have explored negative rates, such as Japan during the 1990's ("The Lost Decade"). However, the outcomes have not been promising, and in general, most economists do not take this option seriously. The "zero lower bound" (ZLB) marks itself as a defining constraint in our work and in the Great Recession. When the U.S. Economy was at the "zero lower bound" in 2008 the Federal Reserve turned a novel alternative now known as "Quantitative Easing" to stimulate the economy (purchasing risky assets that banks had assembled from the mortgage market). For nearly a decade after the recession, the Federal Reserve maintained rates near the zero lower bound, marking the longest period of near-zero interest rates in U.S. history. This unprecedented phenomenon has prompted rigorous academic work from many leading macroeconomists.

We follow suit and study the problem from a mathematical economics perspective, exploring several variations of the optimal interest rate problem where the ZLB binds. In doing so, we hope to add insights to previous literature produced by Eggertson and Woodford, Krugman, Werning, Adam and Billi, Jung et. al, and Christiano et. al.

Werning and Eggerston and Woodford motivate the construction of our model, which is derivative of a standard New Keynesian (NK) model [Wer12] [EW03]. NK models are useful because they account for "price stickiness"–price stickiness is important to embed into a model because prices tend to change slowly with respect to macroeconomic conditions and are a factor of inflation. Another benefit of an NK model is that states and controls are solved in terms of log differences, which approximate percent changes. NK models lend themselves nicely to standard optimization tools involving the Maximum Principle and LQR.

The following is a brief overview of previously published literature on the topic. Christiano et. al explores the outcome for a "Taylor Rule" interest rate policy under varying levels of price stickiness [Chr+11]. Krugman shows how committal (when the Federal Reserve follows through on their previously announced plans for the interest rate) monetary policy can provide a favorable outcome while constrained to the ZLB [Kru98]. Adam and Billi, Jung et. al, and Eggertson and Woodford produce work that shows it is optimal to keep the interest rate at zero even for some time when the ZLB is non-binding [AB06] [Jun+05].

In studying variations of the optimal interest rate problem under this model, we hope to replicate and comment on some insights produced in the literature thus far.

2 Mathematical Representation

As previously mentioned, the optimizing agent is the Federal Reserve. The goal of the Federal Reserve is to maintain suitable levels of inflation and output (typically thought of as Gross Domestic Product). The following is a brief description of the variables and parameters that we will incorporate into our model:

1. $x(t)$ is the output gap (log difference between actual output and the efficient level of output). Using GDP as a unit of measurement, this is $ln(GDP) - ln(GDP_*)$ (where GDP_* is efficient), which can be interpreted as a difference in terms of percent (of GDP_*).

2. $\pi(t)$ denotes the annual rate of inflation (percent).

3. We seek to manipulate (1) and (2) through $i(t)$, the **nominal** interest rate (percent).

4. $r(t)$ is natural rate of interest (percent). For the purposes of this paper, we take the natural rate of interest to be "the real interest rate that would prevail in an efficient, flexible price, outcome with $x(t) = 0$ [efficient output level] throughout" [Wer12].

5. σ is the intertemporal elasticity of substitution (IES). The IES is "a measure of

responsiveness of the growth rate of consumption to the real interest rate" [Hal88]. Literature shows that typical values of $\sigma \in [0.15, 0.75]$. For the purposes of this paper, we let $\sigma = 0.5$.

6. ρ can be taken to be the discount rate, which is empirically shown to be close to 1. We let $\rho = 0.98$.

7. λ reflects how much priority the Fed gives to controlling inflation relative to managing the output gap. We are first interested in a problem where both objectives have equal priority, so we let $\lambda = 1$.

8. $\kappa > 0$ models how flexible prices are. Literature shows $\kappa \in [0.08, 0.20]$, and we set $\kappa = 0.15$.

9. T represents the point in time at which $r(T) > 0$.

With all of the parameters introduced and $x(t)$ and $\pi(t)$ as state variables under the influence of our control $i(t)$, we face the following optimization problem:

$$\frac{1}{2}\int_0^T e^{-\rho t}(x(t)^2 + \lambda\pi(t)^2)dt$$

subject to

$$\dot{x}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t))$$

$$\dot{\pi}(t) = \rho\pi(t) - \kappa x(t)$$

$$i(t) \geqslant 0 \text{ (negative interest rates are not preferred)}$$

and

$$r(t) = \begin{cases} \underline{r} & t < T \\ \overline{r} & t \geqslant T \end{cases}$$

for $\underline{r} < 0 < \overline{r}$, $T < \infty$

3 Solution

Having

$$L(x, \pi, i, t) = \frac{1}{2}(x(t)^2 + \lambda\pi(t)^2)$$

since we are minimizing

$$J[i] = \int_0^\infty e^{-\rho t} L(x, \pi, i, t)dt$$

We can use this modified L (without the discount factor $e^{-\rho t}$) and obtain:

$$H = -L + \mu_x x + \mu_\pi \pi = -\frac{1}{2}x(t)^2 - \frac{1}{2}\lambda\pi(t)^2 + \mu_x\sigma^{-1}(i(t) - r(t) - \pi(t)) + \mu_\pi(\rho\,\pi(t) - \kappa x(t))$$

but we are required to modify our co-state evolution equations

$$\dot{\mu_x}(t) = \rho\mu_x - \frac{\partial H}{\partial x} = x(t) + \kappa\mu_\pi(t) + \rho\ \mu_x(t)$$

$$\mu_x(0) = 0$$

$$\dot{\mu_\pi}(t) = \rho\mu_\pi - \frac{\partial H}{\partial \pi} = \lambda\pi(t) + \sigma^{-1}\mu_x(t)$$

$$\mu_\pi(0) = 0$$

Where co-state initial conditions follow by free initial states. We also get:

$$i(t)\mu_x = 0$$

The conditions on the costate from the maximum principle (for $i(t) > 0$) include:

$$\mu_x(t) \geqslant 0$$

as well as the complementary slackness condition:

$$i(t)\mu_x(t) = 0$$

Assume $i(t) > 0$ over some interval (this implies the zero-bound constraint is not binding, a.k.a. $r(t) > 0$), then $\mu_x(t) = 0$ and $\dot{\mu_x}(t) = 0$ as well on the interval. Then we get a modification of our costate equations on the interval:

$$x(t) = -\kappa\mu_\pi(t)\dot{\mu_\pi}(t) = \lambda\pi(t)$$

and:

$$0 = \dot{x}(t) + \kappa\dot{\mu}_\pi(t) = \dot{x}(t) + \kappa\lambda\pi(t)$$

so

$$-\kappa\lambda\pi(t) = \dot{x}(t) = \sigma^{-1}(i(t) - r(t) - \pi(t))$$

then, by moving the σ over (that is, multiply both sides by σ):

$$i(t) = -\sigma\kappa\lambda\pi(t) + r(t) + \pi(t) = r(t) + (1 - \sigma\kappa\lambda)\pi(t)$$

when $r(t) > 0$, otherwise $i(t) = 0$.

4 Interpretation

4.1 Finite Horizon

We first study alternative versions of the problem by imposing finite time horizons. In these plots, the left hand side shows state and co-state evolutions, and the optimal control is shown on the right subplot. Figures 1 and 3 are solutions under a constant $r(t) > 0$, while figures 2 and 4 are under changing positive and negative $r(t)$.

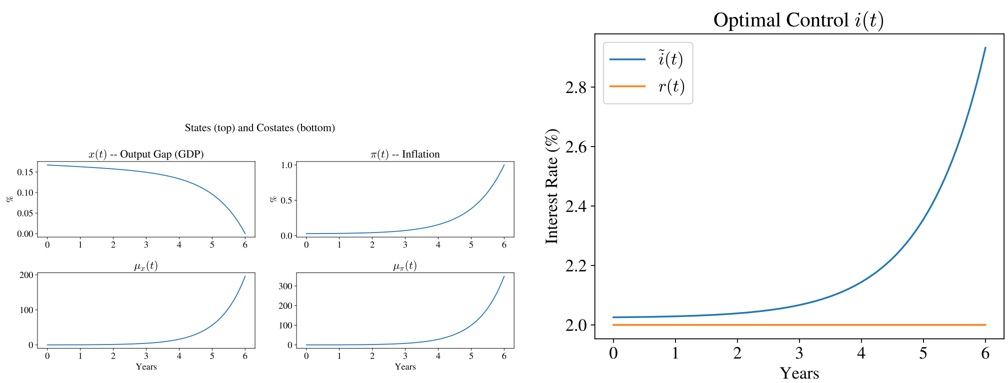

Figure 1: Constant $r(t) = 2$ and $(x(t_f), \pi(t_f)) = (0, 1)$

This first problem demonstrates the optimal path towards some target state specified by the endpoint condition. With this constant $r(t) > 0$, we see that we do not reach the goal until the end of the $t_f = 6$ year period. However, it should be noted that given the setup of the problem, $(x(0), \pi(0))$ was taken to be endogenous. This is not terribly useful, because a policy making agent knows only the initial [present] state and desires to optimize a final state. Nonetheless, these results are informative.

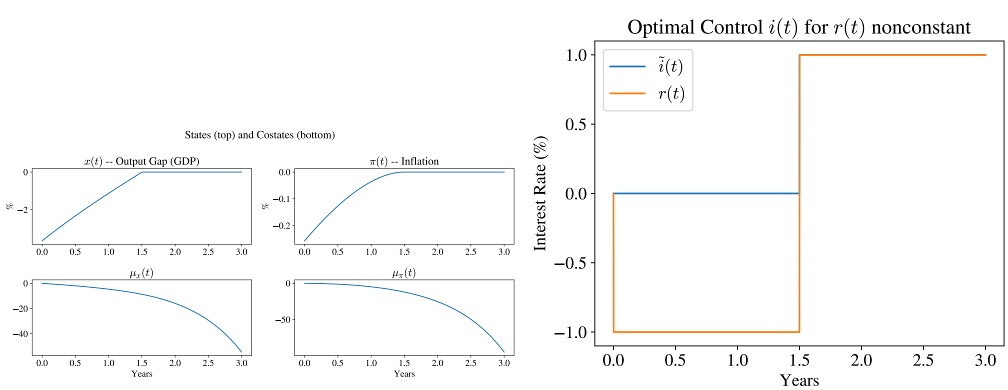

Figure 2: $r(t)$ changing and $(x(t_f), \pi(t_f)) = (0, 0)$

The problem then changes to a varying $r(t)$ that can be positive and negative (we experiment with a different final inflation rate, but both inflation rates are seen as favorable by most Central Banks, and perturbing this parameter helps demonstrate robustness of our observations). This demonstrates the lower bounded-ness of the optimal control throughout the 3 years prior to reaching a target endpoint condition. We notice that the economy reaches the endpoint condition once the control becomes positive, and then setting the control equal to the natural rate $r(t)$ maintains the equilibrium.

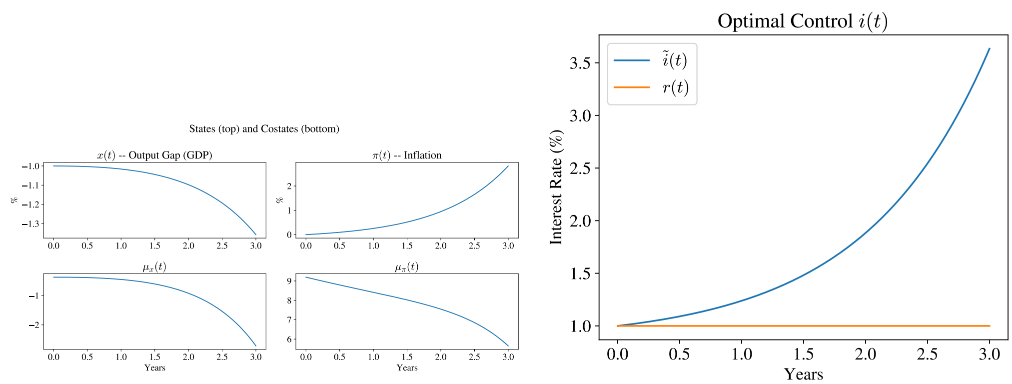

Figure 3: Constant $r(t) = 1$ and $(x(0), \pi(0)) = (-1, 0)$

This problem could potentially model the recovery from a recession (as there is a negative output gap). We see that for a positive constant natural rate, inflation and output gap takeoff in opposite directions and the optimal control is to hike the nominal interest rate. In reality, this is not a sensible response and further demonstrates our learning that our optimization problem is not well-suited for the finite time horizon.

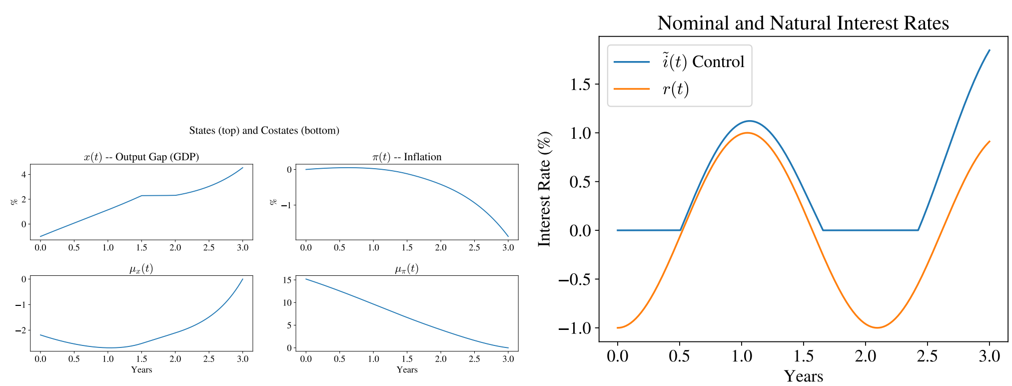

Figure 4: Sinusoidal $r(t)$ and $(x(0), \pi(0)) = (0, 0)$

These figures show the paths of the output gap and inflation after starting at -1% and 0% respectively. When the natural rate is negative, the ZLB constraint binds and we see that $i(t)$ is constantly zero. As the natural interest rate increases towards one, we see the control following $\tilde{i}(t) = r(t) + (1 - \sigma\kappa\lambda)\pi(t)$. Typical behavior of the natural rate likely does not follow the smooth path of a sinusoid, however it is possible that we see this degree of fluctuation. We also observe that in contrast to Adam and Billi, Jung et. al, and Eggertson and Woodford—which state that the control will be 0 even for times when $r(t)$ is not negative—the optimal control is positive for negative $r(t)$.

4.2 Infinite Horizon

We can solve the infinite horizon problem by finding the solution P of the Algebraic Ricatti Equation given by $PA + A^TP + Q - PBR{-}1B^TP = 0$ where our cost functional is defined by

$$\int_0^\infty (\mathbf{x}^T Q\mathbf{x} + i^T Ri)dt$$

where $\mathbf{x} = (x(t), \pi(t))$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$ and $R = .0001$. and our state evolution is defined by

$$\mathbf{x}' = A\mathbf{x} + Bi$$

with $A = \begin{bmatrix} 0 & -\sigma^{-1} \\ -\kappa & \rho \end{bmatrix}$, and $B = \begin{bmatrix} -\sigma^{-1} \\ 0 \end{bmatrix}$.

We select a small value for R to reflect the general belief that there is not much cost to changing $i(t)$, and to satisfy the requirement of invertibility of R for existence of a solution. We highlight that this is no longer a bang-bang problem in terms $i(t)$ under this modification.

Since this assumes an LQR solution of $x' = Ax + Bu$ ignoring the $f(t) = (-\sigma^{-1}r(t), 0)$, we must now estimate our optimal control i by $i(t) = B^{-1}B\tilde{i} - B^{-1}f$. However, our f(t) only changes the first dimension x(t) - the output gap, so we can take $B = \sigma^{-1}$, and $f(t) = -\sigma^{-1}r(t)$. So it follows that $i(t) = B^{-1}B\tilde{i} - B^{-1}f = \tilde{i} - r(t)$.

Below are plots showing 10 years from the solution to the infinite horizon problem with constant and non-constant r.

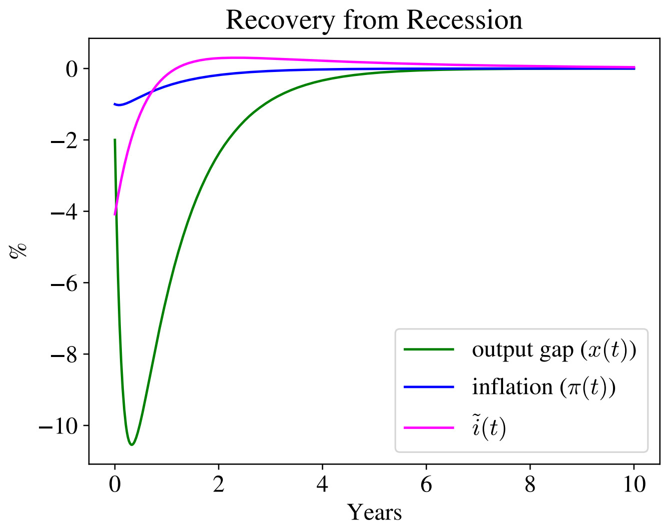

Figure 5: 10 year solution to the LQR infinite horizon problem with constant r. We let $r(t) = r = 6\%$, and this solution allows for negative interest.

We not yet aware of how to approach an LQR problem where a bound is imposed on the control, and note that this result might not be unrealistic for a Central Bank willing to flirt with a negative $i(t)$.

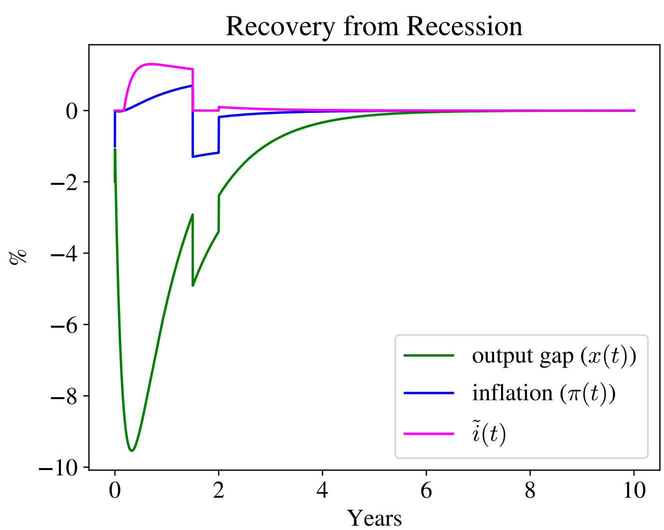

Figure 6: 10 year solution to the LQR infinite horizon problem with non-constant r. We let $r(t)$ be a non-continuous piecewise function simulating sudden events and shifts in the market.

5 Conclusion and Extension

In this paper we have been able to explore solutions to the optimal control of Federal Reserve's monetary policy aimed at stabilizing inflation and economic output. We formulated an optimization problem that was solved using Pontryagin's Maximum Principle to determine the optimal interest rate policy for various conditions over finite and infinite time horizons. Our results show that optimal monetary policy can be used to effectively balance the trade-off between inflation and economic output in certain scenarios. With respect to pertinent literature on the topic, we also found that—as shown in figure 4—the optimal control is positive for times when $r(t) < 0$, which contradicts previous findings in the literature and may be a result of the set-up of our problem. In future work, we hope to explore the effects of the sub-optimal "Taylor Rule" for $i(t)$ in comparison to the optimal control.

The reality is that, beyond the many quantifiable and unquantifiable factors that play into national macroeconomic behavior (that are not included in our modeling), there are also a variety of controls that can be placed/tuned by the Federal Reserve and related policy makers. One of those suggested in the aforementioned literature is the role of government spending and is discussed in detail in the appendix.

A Appendix

We will denote government spending as g. We then get, with the inclusion of a constant Γ to represent taxation, the control problem is to minimize:

$$\frac{1}{2}\int_0^\infty e^{-\rho t}(x(t)^2 + \lambda\pi(t)^2 + \eta g(t)^2)dt$$

subject to:

$$\dot{x}(t) = (1-\Gamma)\dot{g}(t) + \sigma^{-1}(i(t) - r(t) - \pi(t))$$
$$\dot{\pi}(t) = \rho\pi(t) - \kappa x(t)$$
$$i(t) \geqslant 0$$
$$x(0), \pi(0) \text{ free}$$

Which gives the Hamiltonian:

$$H = -\frac{1}{2}x(t)^2 - \frac{1}{2}\lambda\pi(t)^2 - \frac{1}{2}\eta g(t)^2 + \mu_x\left((1-\Gamma)\dot{g}(t)\sigma^{-1}(i(t) - r(t) - \pi(t))\right) + \mu_\pi(\rho\pi(t) - \kappa x(t))$$

and the same co-state evolution equations as before:

$$\dot{\mu_x}(t) = \rho\mu_x - \frac{\partial H}{\partial x} = x(t) + \kappa\mu_\pi(t) + \rho\mu_x(t)$$

$$\dot{\mu_\pi}(t) = \rho\mu_\pi - \frac{\partial H}{\partial \pi} = \lambda\pi(t) + \sigma^{-1}\mu_x(t)$$

Which we solve by seeing that the control can be cast as $\dot{g}(t)$, which we now call $u(t)$ (we designate $g(t)$ as a state variable), whose evolution equation is the previously mentioned:

$$\dot{g}(t) = u(t)$$

We then add $\mu_g(t)u(t)$ to the Hamiltonian and get the co-state evolution:

$$\dot{\mu}_g(t) = \rho\mu_g(t) + \eta g(t)$$

It can be shown that in cases in which $i(t) > 0$, $u(t) = 0$ is the optimal control. Thus the relevant cases for this extension are when $i(t) = 0$, which means that the Hamiltonian becomes linear in our control $u(t)$ and maximizing it will result in a bang-bang problem with coefficient:

$$\mu_x(t)(1 - \Gamma) + \mu_g(t)$$

In future work we aim to offer a thorough study of this problem as it pertains to interesting macroeconomics questions of fiscal (government stimulus spending) and monetary policy (interest rates) coordination.

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